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Remainder Formulas and Optimal Points for Exponential and Trigonometric Interpolation

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1. INTRODUCTION

Let $\phi_0(x)$, $\phi_1(x),\ldots, \phi_n(x)$ be arbitrary, but linearly independent functions, and $f(x)$ a function given in analytical or tabular form. The linear interpolation problem consists in determining coefficients $\gamma_0, \gamma_1, \ldots, \gamma_n$ such that the function

$$\Phi(x) = \sum_{i=0}^{n} \gamma_i \phi_i(x) \tag{1.1}$$

satisfies the equalities

$$\Phi(x_i) = f(x_i), \qquad i = 0, 1,\ldots, n, \tag{1.2}$$

where the given abcissae x_i are the so-called interpolation points.

Assume that these points lie in a finite interval $[a, b]$, and that $f(x)$, $\phi_0(x),\ldots, \phi_n(x)$ are defined on that interval. It is well known [1] that the linear interpolation problem has a unique solution on $[a, b]$ if and only if the system $\{\phi_0, \phi_1, \ldots, \phi_n\}$ satisfies the Haar condition on $[a, b]$, i.e., $\det(\phi_i(x_j)) \neq 0$ for every cloice of $n+1$ points $x_0, x_1, \ldots, x_n$ from $[a, b]$. Equivalent with this condition is the statement that every function of the form (1.1) has at most n zeros in that interval [2].

The main examples of linear interpolation types allowing a unique solution are

(i) polynomial interpolation, where

$$\phi_i(x) = x^i, \quad i = 0, 1,\ldots, n; \qquad a, b \text{ arbitrary;} \tag{1.3.a}$$

(ii) exponential interpolation [3], where in its most general form,

$$\phi_i(x) = e^{\beta_i x}, \qquad i = 0, 1, \dots, n; \qquad a, b \text{ and the } \beta_i \text{ arbitrary;} \tag{1.3.b}$$

(iii) trigonometric interpolation [4], where

$$\phi_0(x) = 1, \quad \phi_{2i-1}(x) = \cos ix, \quad \phi_{2i}(x) = \sin ix, \quad i = 1, 2, \dots, N; \quad n = 2N; \quad b - a = 2\pi. \tag{1.3.c}$$

The more general trigonometric system

$$\phi_{2i}(x) = \cos \beta_i x, \quad \phi_{2i+1}(x) = \sin \beta_i x, \quad i = 0, 1, \dots, N, \quad n = 2N + 1, \tag{1.3.d}$$

does not always satisfy the Haar condition. Other trigonometric Haar systems are [5]

$$\left.\begin{array}{l} \phi_i(x) = \sin ix, \\ \phi_0(x) = 1, \qquad \phi_i(x) = \cos ix, \end{array}\right. \quad i = 1, 2, \dots, n; \quad b - a < 2\pi. \tag{1.3.e}$$

In interpolation theory it is important to know the remainder $R_{n+1}(x) = f(x) - \Phi(x)$. If the functions $f(x)$, $\phi_0(x), \dots, \phi_n(x)$ are $n + 1$ times differentiable on $[a, b]$, then Petersson's remainder formula holds [6]

$$R_{n+1}(x) = L_{n+1}[f(\xi)] \cdot h(x), \tag{1.4}$$

where $x, \xi \in [a, b]$; the differential operator L_{n+1} is defined by

$$L_{n+1}[g(x)] = \begin{vmatrix} \phi_0(x) & \cdots & \phi_n(x) & g(x) \\ \cdot & \cdots & \cdot & \cdot \\ \phi_0^{(n+1)}(x) & \cdots & \phi_n^{(n+1)}(x) & g^{(n+1)}(x) \end{vmatrix} \Big/ W(x), \tag{1.5}$$

where $W(x)$ is the Wronskian

$$W(x) = \begin{vmatrix} \phi_0(x) & \cdots & \phi_n(x) \\ \cdot & \cdots & \cdot \\ \phi_0^{(n)}(x) & \cdots & \phi_n^{(n)}(x) \end{vmatrix}; \tag{1.6}$$

further $h(x)$ is the solution of the differential equation

$$L_{n+1}[h(x)] = 1, \qquad h(x_i) = 0, \qquad i = 0, 1, \dots, n. \tag{1.7}$$

We call *optimal interpolation points* the zeros $x_0, x_1, \dots, x_n$ of $h(x)$ which are so chosen that this function is the minimax approximation to the null function on $[a, b]$. The problem of finding optimal interpolation points is solved for the case of polynomial interpolation (1.3.a). Then clearly

$L_{n+1}[f(\xi)] = f^{(n+1)}(\xi)$, and $L_{n+1}[h(x)] = h^{(n+1)}(x) = 1$, with $h(x_i) = 0$, $i = 0, 1,\ldots, n$. The solution of the latter equation is

$$h(x) = \frac{w(x)}{(n+1)!}, \qquad w(x) = \prod_{i=0}^{n} (x - x_i). \tag{1.8}$$

If $a = -1$, $b = 1$, the optimal x_i are the zeros of the polynomial $w(x) = 2^{-n} T_{n+1}(x)$, where $T_{n+1}(x)$ is the Chebyshev polynomial of the first kind. The zeros x_i, extremal points $\bar{x}_i$ and deviation d of $w(x)$ are then given by [7]

$$x_i = -\cos(2i+1)\pi/(2n+2), \quad i = 0, 1,\ldots, n; \tag{1.9}$$

$$\bar{x}_i = -\cos i\pi/(n+1), \qquad i = 0, 1,\ldots, n+1; \quad d = 2^{-n}. \tag{1.10}$$

The linear transformations

$$x = [(b-a)/2]u + (b+a)/2; \qquad u = (2x - b - a)/(b - a);$$
$$a \leqslant x \leqslant b; \qquad -1 \leqslant u \leqslant 1 \tag{1.11}$$

map the intervals $[a, b]$ and $[-1, +1]$ onto each other. Since the linear space spanned by the system (1.3.a) is invariant with respect to the first transformation (1.11), optimal interpolation points are also known for an arbitrary interval $[a, b]$.

In the present paper Petersson's remainder formula will be used to develop a similar theory of optimal interpolation for some types of exponential and trigonometric interpolation. Results will be compared with the polynomial case.

2. Remainder Formulas

We consider the systems (1.3.b) and (1.3.d) and assume firstly that all β_i differ from zero. (The β_i must be such that the system (1.3.d) satisfies the Haar condition.) We write the differential equation (1.7) in the following explicit form:

$$\sum_{k=0}^{n+1} c_k(x)\, h^{(k)}(x) = 1, \qquad h(x_i) = 0, \qquad i = 0, 1,\ldots, n. \tag{2.1}$$

By virtue of (1.5), (1.6) we find for exponential interpolation (1.3.b) that

$$c_0(x) = (-1)^{n+1} \prod_{i=0}^{n} \beta_i .$$

This is a constant and hence a particular solution of (2.1) is $(-1)^{n+1}/(\beta_0\beta_1\cdots\beta_n)$. Since every function of the form $\sum_{i=0}^{n} \gamma_i e^{\beta_i x}$ is a solution of the homogeneous equation (2.1), the total solution of this equation is

$$h(x) = \sum_{i=0}^{n} \gamma_i e^{\beta_i x} + (-1)^{n+1}/(\beta_0\beta_1 \cdots \beta_n), \tag{2.2}$$

where the constants γ_i are fixed by the conditions $h(x_j) = 0$, $j = 0, 1,..., n$.

After a more elaborate calculation we find for trigonometric interpolation (1.3.d) that

$$c_0(x) = \prod_{i=0}^{N} \beta_i^2;$$

the solution of (2.1), consequently, is

$$h(x) = \sum_{i=0}^{N} (\gamma_i \cos \beta_i x + \delta_i \sin \beta_i x) + (\beta_0\beta_1 \cdots \beta_N)^{-2}, \tag{2.3}$$

where the γ_i, δ_i are such that $h(x_j) = 0$, $j = 0, 1,..., n$.

Secondly, we assume that $\beta_0 = 0$. Then it follows from (1.5) and (1.6) that $c_0(x) = 0$. Furthermore, for the exponential case,

$$c_1(x) = (-1)^n \prod_{i=1}^{n} \beta_i,$$

and, hence,

$$h(x) = \gamma_0 + \sum_{i=1}^{n} \gamma_i e^{\beta_i x} + \frac{(-1)^n}{\beta_1\beta_2 \cdots \beta_n} x. \tag{2.4}$$

For the trigonometric case we now have $\phi_1 = 0$; therefore, we redefine the system (1.3.d) as

$$\phi_0(x) = 1, \qquad \phi_{2i-1}(x) = \cos \beta_i x, \qquad \phi_{2i}(x) = \sin \beta_i x,$$
$$i = 1, 2,..., N; \qquad n = 2N. \tag{2.5}$$

Then Eqs. (1.5), (1.6), and (2.1) yield

$$c_1(x) = \prod_{i=1}^{N} \beta_i^2;$$

consequently,

$$h(x) = \gamma_0 + \sum_{i=1}^{N} (\gamma_i \cos \beta_i x + \delta_i \sin \beta_i x) + (\beta_1\beta_2 \cdots \beta_N)^{-2} x. \tag{2.6}$$

In practice the numbers β_i are chosen as follows.

(i) Exponential case. We take

$$\beta_0 = 0; \quad \beta_{2i-1} = i, \quad \beta_{2i} = -i, \quad i = 1, 2,..., N; \quad n = 2N. \tag{2.7}$$

Then the function $h(x)$, given by (2.4), reduces to

$$h_e(x) = \gamma_0 + \sum_{i=1}^{N} (\gamma_{2i-1}e^{ix} + \gamma_{2i}e^{-ix}) + (-1)^N x/(N!)^2. \tag{2.8}$$

The determination of the coefficients γ_i from the equations $h_e(x_j) = 0$ $j = 0, 1,..., n$, is a problem of exponential interpolation of the function $g(x) = (-1)^{N+1}x/(N!)^2$, the solution $h_1(x)$ of which is known to be [6]

$$h_1(x) = \sum_{k=0}^{2N} \frac{E_k(x)}{E_k(x_k)} g(x_k), \qquad E_k(x) = \prod_{j \neq k}^{0,2N} \sinh\left(\frac{x - x_j}{2}\right);$$

hence,

$$h_e(x) = \frac{(-1)^{N+1}}{(N!)^2} w_e(x), \qquad w_e(x) = \sum_{k=0}^{2N} \frac{E_k(x)}{E_k(x_k)} x_k - x. \tag{2.9}$$

We can generalize the choice (2.7) to

$$\beta_0 = 0; \quad \beta_{2i-1} = i\beta, \quad \beta_{2i} = -i\beta, \quad i = 1, 2,..., N; \quad \beta > 0. \tag{2.10}$$

The remainder formula (2.9) is modified accordingly

$$h_e(x) = \frac{(-1)^{N+1}}{(\beta^N N!)^2} w_e(x), \qquad w_e(x) = \sum_{k=0}^{2N} \frac{E_k(x)}{E_k(x_k)} x_k - x,$$
$$E_k(x) = \prod_{j \neq k}^{0,2N} \sinh\left[\beta\left(\frac{x - x_j}{2}\right)\right]. \tag{2.11}$$

(ii) Trigonometric case. We take the system (1.3.c), i.e.,

$$\beta_0 = 0; \quad \beta_i = i, \quad i = 1, 2,..., N; \quad n = 2N; \quad b - a \leq 2\pi. \tag{2.12}$$

It can be shown along similar lines as in the previous case that the remainder becomes

$$h_t(x) = -\frac{1}{(N!)^2} w_t(x), \qquad w_t(x) = \sum_{k=0}^{2N} \frac{S_k(x)}{S_k(x_k)} x_k - x,$$
$$S_k(x) = \prod_{j \neq k}^{0,2N} \sin\left(\frac{x - x_j}{2}\right). \tag{2.13}$$

Again we can generalize (2.12) to

$$\beta_0 = 0; \quad \beta_i = i\beta, \quad i = 1, 2,..., N; \quad \beta > 0; \quad \beta(b - a) \leq 2\pi. \tag{2.14}$$

We then obtain for the remainder (2.6)

$$h_l(x) = \frac{1}{(\beta^N N!)^2} w_l(x), \qquad w_l(x) = \sum_{k=0}^{2N} \frac{S_k(x)}{S_k(x_k)} x_k - x,$$
$$S_k(x) = \prod_{j \neq k}^{0,2N} \sin\left[\beta\left(\frac{x - x_j}{2}\right)\right]. \tag{2.15}$$

Hamming [11, pp. 280–283] gives a remainder formula for trigonometric interpolation of the type (1.3.c) which is valid for equidistant nodes. It can be shown that the application of Petersson's formula to systems of the type (1.3.e) does not lead to analogous simple expressions for the remainder. However, an expression for the remainder of generalized interpolation including the type (1.3.c) (and its hyperbolic analog) has been given by Newbery [12]. The quantity corresponding to our $h(x)$ there is

$$\frac{\phi_0(x)\, w(x)}{(n+1)!}, \qquad w(x) = \prod_{i=0}^{n} (\cos x - \cos x_i),$$

with $\phi_0(x) = \sin x$ or $= 1$. It follows at once from this and Eq. (1.9) that the optimal interpolation points on the interval $[0, \pi]$ are

$$x_i = (2i+1)\pi/(2n+2), \qquad i = 0, 1, \ldots, n.$$

The systems (1.3.e) will not further be treated here.

3. Calculation of Optimal Interpolation Points

We like to determine the optimal interpolation points for exponential interpolation of the type (2.10) on an arbitrary interval $[a, b]$, and for trigonometric interpolation of the type (2.14) on an interval $[a, b]$ with $\beta(b - a) < 2\pi$. For reasons of convenience we wish to perform the computations on the interval $[-1, 1]$. However, the spaces spanned by the systems corresponding to (2.10) and (2.14) are not invariant with respect to the linear transformations (1.11); these systems are changed into

$$\{1, e^{\alpha u}, e^{-\alpha u}, \ldots, e^{N\alpha u}, e^{-N\alpha u}\},$$
$$\{1, \cos \alpha u, \sin \alpha u, \ldots, \cos N\alpha u, \sin N\alpha u\}, \qquad \alpha = \beta(b - a)/2,$$

where u is the variable in the interval $[-1, 1]$. Therefore, the problem of determining optimal points on an arbitrary interval can be solved without loss of generality by regarding β and α as parameters (mostly $\beta = 1$), and by solving the same problem on the interval $[-1, 1]$; we have $0 < \alpha < \infty$ in the exponential case and $0 < \alpha < \pi$ in the trigonometric case.

It immediately follows from (2.11) and (2.15) (where x and β have been replaced by u and α, respectively) that we have to determine functions

$$\Phi_e(u) = \sum_{k=-N}^{N} \gamma_k e^{\alpha k u} \qquad \text{and} \qquad \Phi_t(u) = \gamma_0 + \sum_{k=1}^{N} (\gamma_k \cos \alpha k u + \delta_k \sin \alpha k u)$$

that are minimax approximations on $[-1, 1]$ to the function $f(u) = u$. Since the latter is odd, so are Φ_e and Φ_t ; they, consequently, have the forms

$$\Phi_e(u) = \sum_{k=1}^{N} a_k \sinh \alpha k u; \qquad \Phi_t(u) = \sum_{k=1}^{N} b_k \sin \alpha k u. \tag{3.1}$$

The optimal interpolation points are the zeros of the functions

$$w_e(u) = \Phi_e(u) - u; \qquad w_t(u) = \Phi_t(u) - u. \tag{3.2}$$

The (at least) $2N + 2$ extremal points of $w_e(u)$ and $w_t(u)$ are 2×2 symmetrically situated with respect to the origin. So $w_e(u)$ and $w_t(u)$ have (at least) $N + 1$ extremal points, and, hence, (at least) N zeros in the interval $(0, 1]$. The extremal points are zeros of the functions

$$w_e'(u) = \alpha \sum_{k=1}^{N} k a_k \cosh \alpha k u - 1; \qquad w_t'(u) = \alpha \sum_{k=1}^{N} k b_k \cos \alpha k u - 1. \tag{3.3}$$

By the substitutions $z = \cosh \alpha u$ and $z = \cos \alpha u$, respectively, these functions are transformed into polynomials in z of degree N and, hence, have at most N zeros in $(0, 1]$. Consequently, there must be an extremal point at $u = 1$. Furthermore, $w_e(u)$ and $w_t(u)$ have precisely N zeros in $(0, 1]$; between two adjacent extreme values these functions are strictly monotonic.

In view of these remarks the minimax approximations (3.1) can be computed effectively by means of Remez' second algorithm [7]. Let us denote by e_i, t_i, $i = 1, 2, \ldots, N$, the positive zeros of $w_e(u)$ and $w_t(u)$, respectively, the extremal points by $\bar{e}_i$, $\bar{t}_i$, $i = 1, 2, \ldots, N + 1$ (with $\bar{e}_{N+1} = \bar{t}_{N+1} = 1$), and the deviations by d_e, d_t. These quantities, together with the coefficients a_k, b_k, must be calculated from the equations

$$\sum_{k=1}^{N} a_k \sinh \alpha k \bar{e}_i - \bar{e}_i = (-1)^i d_e, \qquad i = 1, 2, \ldots, N + 1, \tag{3.4.a}$$

$$\alpha \sum_{k=1}^{N} k a_k \cosh \alpha k u - 1 = 0, \tag{3.4.b}$$

$$\sum_{k=1}^{N} a_k \sinh \alpha k u - u = 0, \tag{3.4.c}$$

and

$$\sum_{k=1}^{N} b_k \sin \alpha k \bar{t}_i - \bar{t}_i = (-1)^i d_t, \qquad i = 1, 2, \ldots, N+1, \tag{3.5.a}$$

$$\alpha \sum_{k=1}^{N} k b_k \cos \alpha k u - 1 = 0, \tag{3.5.b}$$

$$\sum_{k=1}^{N} b_k \sin \alpha k u - u = 0. \tag{3.5.c}$$

The remainder $R_{2N+1}(u)$ of the interpolation types under consideration consists of two factors (see (1.4)). The first, $L_{2N+1}[f(\xi)]$, depends on the given function $f(u)$ and on α and is in general unestimable. The second, $h_e(u)$ or $h_t(u)$, given by (2.11) or (2.15) with β and x replaced by α and u, has been minimized by an appropriate choice of the interpolation points. When putting

$$\epsilon_e = \frac{(-1)^{N+1} d_e}{(\alpha^N N!)^2}, \qquad \epsilon_t = \frac{-d_t}{(\alpha^N N!)^2}, \tag{3.6}$$

i.e., $|\epsilon_e| = \max |h_e(u)|$, $|\epsilon_t| = \max |h_t(u)|$, we may say that the accuracy of optimal exponential or trigonometric interpolation ultimately depends on ϵ_e and ϵ_t.

Following Nitsche [8] we call e_i, t_i, $\bar{e}_i$, $\bar{t}_i$, d_e, d_t, ϵ_e, ϵ_t the *Chebyshev quantities* of our approximation problem. Tables I, II, and III contain numerical values of these quantities for $N = 3$ and $\alpha = \frac{1}{2}$, 1, π. Table IV

TABLE I

$\alpha = 0.5$

i	$\bar{e}_i$	e_i	$\bar{t}_i$	t_i
1	0.22399	0.43632	0.22105	0.43143
2	0.62612	0.78392	0.62082	0.77970
3	0.90213	0.97526	0.89977	0.97459
	$d_e = 0.16818 \times 10^{-5}$		$d_t = -0.18090 \times 10^{-5}$	
	$\epsilon_e = 0.29898 \times 10^{-5}$		$\epsilon_t = 0.32160 \times 10^{-5}$	

gives the corresponding quantities for the case of polynomial interpolation. We denote by p_i and $\bar{p}_i$, $i = 1, 2, \ldots, N$, the positive zeros and extremal points of $T_{2N+1}(u)$, which are given by (see (1.9) and (1.10))

$$p_i = \sin i\pi/(2N+1), \quad \bar{p}_i = \cos(N-i+1)\pi/(2N+1), \quad i = 1, 2, \ldots, N. \tag{3.7}$$

TABLE II

$\alpha = 1$

i	$\bar{e}_i$	e_i	$\bar{t}_i$	t_i
1	0.22837	0.44353	0.21662	0.42396
2	0.63378	0.78991	0.61260	0.77300
3	0.90542	0.97618	0.89597	0.97349
	$d_e = 0.96785 \times 10^{-4}$		$d_t = -0.12960 \times 10^{-3}$	
	$\epsilon_e = 0.26885 \times 10^{-5}$		$\epsilon_t = 0.36000 \times 10^{-5}$	

TABLE III

$\alpha = \pi$

i	$\bar{e}_i$	e_i	$\bar{t}_i$	t_i
1	0.27780	0.51744	0.16316	0.32589
2	0.70389	0.83933	0.48755	0.64678
3	0.93045	0.98289	0.80000	0.93443
	$d_e = 0.32546 \times 10^{-1}$		$d_t = -1$	
	$\epsilon_e = 0.94038 \times 10^{-6}$		$\epsilon_t = 0.28893 \times 10^{-4}$	

TABLE IV

i	$\bar{p}_i$	p_i
1	0.22252	0.43388
2	0.62349	0.78183
3	0.90097	0.97493
	$d_p = 0.15625 \times 10^{-1}$	
	$\epsilon_p = 0.31002 \times 10^{-5}$	

The maxima d_p and ϵ_p of $w(u)$ and $h(u)$ (see (1.10) and (1.8)) are

$$d_p = 2^{-2N}, \qquad \epsilon_p = d_p/(2N+1)\,! \,. \tag{3.8}$$

The Chebyshev quantities are functions of α, defined in the ranges $(0, \infty)$ and $(0, \pi)$. It is seen from (3.1) and (3.2) that a sign changement of α only causes a sign changement of the coefficients a_k, b_k. Hence, we can extend the definition interval of these functions to $(-\infty, \infty)$ and $[-\pi, \pi]$ with exception of the point $\alpha = 0$. The value $\alpha = 0$ makes no sense from the interpolatory point of view, but we can define formally a Chebyshev quantity for $\alpha = 0$ as the limiting value of that quantity for $\alpha \to 0$. It now appears from (3.1), (3.2), (3.4), and (3.5) that the coefficients a_k, b_k are odd functions of α, while the Chebyshev quantities are even in α.

Our approximation problem is equivalent to approximating the function $f(y) = y$ on the interval $[-\alpha, \alpha]$ by a function of the form $\sum_{k=1}^{N} \gamma_k \sinh ky$ or $\sum_{k=1}^{N} \gamma_k \sin ky$. Let us use the same symbols, but marked with a star, for the Chebyshev quantities and function coefficients related to the interval $[-\alpha, \alpha]$. It is immediately clear from (3.4) and (3.5) that a starred quantity equals α times the corresponding unstarred quantity. It is known [7] that the starred quantities depend continuously on α. In the polynomial case we also have $p_i^* = \alpha p_i$, $\bar{p}_i^* = \alpha \bar{p}_i$, but $d_p^* = \alpha^{2N+1} d_p$.

The quantities a_k, b_k, $k = 1, 2, \ldots, N$, d_e, and d_t have the following sign properties.

THEOREM A.

$$\operatorname{sgn} a_k = \operatorname{sgn} b_k = (-1)^{k+1}, \qquad k = 1, 2, \ldots, N; \qquad d_e > 0,$$

$\operatorname{sgn} d_t = (-1)^N$, *where* sgn *denotes the signum function.*

Proof. The quantities $\bar{e}_i$ are the positive roots of Eq. (3.4.b), which after the substitution $v = e^{\alpha u}$ can be written as

$$v^{-N}\Big[Na_N v^{2N} + (N-1)\, a_{N-1} v^{2N-1} + \cdots + a_1 v^{N+1} - \frac{2}{\alpha} v^N + a_1 v^{N-1} + \cdots + Na_N\Big] = 0.$$

This equation has $2N$ positive roots $e^{\pm\alpha\bar{e}_i}$. Hence, by virtue of Descartes' rule the sequence of coefficients must have the maximum number of sign variations, whence it follows that $\operatorname{sgn} a_k = (-1)^{k-1}$. Consequently,

$$\operatorname{sgn} w_e'(u) = (-1)^{N+1} \qquad \text{for} \quad u > \bar{e}_N .$$

The last equation (3.4.a) then shows that $d_e > 0$.

The proof for the trigonometric case is more complicated. Equation (3.5.b) can be reduced to a polynomial equation $P(v) = \sum_{k=0}^{N} \alpha c_k v^k = 0$ by means of the transformation $v = \cos \alpha u$. The latter equation has N positive roots $\cos \alpha \bar{t}_i$ if $\alpha \leqslant \pi/2$. We first show that the theorem is true for $\alpha < \pi/2$. Using the relation (see [9, formula 403.3])

$$\cos kx = \sum_{j=0}^{[k/2]} (-1)^j \frac{k(k-j-1)(k-j-2)\cdots(k-2j+1)}{j!} 2^{k-2j-1} \cos^{k-2j} x,$$

we find for the coefficients c_k the following expressions:

$$c_{N-2j} = 2^{N-2j-1}[(N-2j)\, b_{N-2j} - C_{N-2j}], \qquad j = 0, 1, \ldots, [N/2], \qquad (3.9.a)$$

where

$$C_{N-2j} = \sum_{l=1}^{j} (-1)^{l-1} \frac{(N-2j+2l)(N-2j+l-1)\cdots(N-2j+1)}{l!} \\ \times (N-2j+2l)\, b_{N-2j+2l}\,, \tag{3.9.b}$$

and

$$c_{N-2j+1} = 2^{N-2j}[(N-2j+1)\, b_{N-2j+1} - C_{N-2j+1}],$$
$$j = 1, 2,\ldots, [(N-1)/2], \tag{3.10.a}$$

where

$$C_{N-2j+1} = \sum_{l=1}^{j-1} (-1)^{l-1} \frac{(N-2j+2l+1)(N-2j+l)\cdots(N-2j+2)}{l!} \\ \times (N-2j+2l+1)\, b_{N-2j+2l+1}\,. \tag{3.10.b}$$

In (3.9) and (3.10), if $N = 2j$ or $N = 2j-1$, the expression $0.b_0$ must be replaced by $-1/\alpha$; further $\sum_{l=1}^{0} = 0$. The coefficients c_k have alternating signs. Suppose $b_N > 0$; then $c_{N-2j} > 0$, $c_{N-2j+1} < 0$, $b_{N-1} < 0$ and, by virtue of (3.9.a) and (3.10.a),

$$(N-2j)\, b_{N-2j} > C_{N-2j}\,; \qquad (N-2j+1)\, b_{N-2j+1} < C_{N-2j+1}\,. \tag{3.11}$$

From (3.9.b) and the first inequality (3.11) we can derive the following chain of inequalities:

$$\begin{aligned}
(N-2j)\, b_{N-2j} &> C_{N-2j} \\
&= \sum_{k=2}^{j} (-1)^{k-2} \frac{\begin{bmatrix}(N-2j+2k)(N-2j+k+1) \\ (N-2j+k-1)\cdots(N-2j+2)\end{bmatrix}}{1!(k-2)!k} \\
&\qquad \times (N-2j+2k)\, b_{N-2j+2k} \\
&> \sum_{k=3}^{j} (-1)^{k-3} \frac{\begin{bmatrix}(N-2j+2k)(N-2j+k-2)(N-2j+k+1) \\ (N-2j+k-1)\cdots(N-2j+3)\end{bmatrix}}{2!(k-3)!k} \\
&\qquad \times (N-2j+2k)\, b_{N-2j+2k} \\
&> \cdots > \sum_{k=l}^{j} (-1)^{k-l} \frac{\begin{bmatrix}(N-2j+2k)(N-2j+k+l-1) \\ (N-2j+k+l-2)\cdots(N-2j+k+1)\end{bmatrix}}{(l-1)!(k-l)!k} \\
&\qquad \times (N-2j+k-1)\cdots(N-2j+l)(N-2j+2k)\, b_{N-2j+2k} \\
&> \cdots > \frac{N(N-1)\cdots(N-j+1)}{(j-1)!\,0!\,j} Nb_N = \binom{N}{j} Nb_N > 0,
\end{aligned}$$
$$j = 0, 1,\ldots, [N/2]. \tag{3.12}$$

In the same way it follows from (3.10.b) and the second inequality (3.11) that

$$(N - 2j + 1)\, b_{N-2j+1} < C_{N-2j+1} < \binom{N-1}{j-1} b_{N-1} < 0,$$

$$j = 1, 2, \ldots, [(N+1)/2]. \quad (3.13)$$

The inequalities (3.12) and (3.13) are reversed if $b_N < 0$. It follows that the coefficients b_k have alternating signs. Suppose now N even and $b_N > 0$. Then, from (3.9.a) and (3.9.b), $c_0 = -\frac{1}{2}(1/\alpha + C_0) > 0$; this is not possible since, by (3.12), also $C_0 > 0$. Suppose next N odd and $b_N < 0$; then $b_{N-1} > 0$ and $c_0 > 0$ which, again, is impossible since by virtue of (3.13) (with the inequality sign reversed), $C_0 > 0$. Hence, $\operatorname{sgn} b_N = (-1)^{N+1}$ and, consequently, $\operatorname{sgn} b_k = (-1)^{k+1}$.

Returning to the general case that $0 < \alpha \leq \pi$, it is impossible that a coefficient b_l equals zero for a certain value of α; indeed, the system $\{1, \cos \alpha u, \ldots, \cos(l-1)\alpha u, \cos(l+1)\alpha u, \ldots, \cos N\alpha u\}$ is a Haar system on the interval $(0, \pi]$ and, hence, every function of the form $\sum_{k \neq l}^{0,N} c_k \cos \alpha k u - 1$ has at most $N - 1$ positive zeros, contradicting the fact that the function (3.5.b) has N positive zeros. Since the coefficients b_k are continuous functions of α (except perhaps in $\alpha = 0$), they never change sign.

Finally, in order to determine the sign of d_l we observe that the substitution $v = \cos \alpha u$ transforms the u-interval $[0, 1]$ into the v-interval $[\cos \alpha, 1]$, whereby $u = 0$ corresponds with $v = 1$ and $u = 1$ with $v = \cos \alpha$. For the roots $v_i = \cos \alpha \bar{t}_i$ of the transformed equation $P(v) = 0$ we have $v_i > v_{i+1}$. Hence, $\{P(v)\}_{v > v_1}$ is positive if $b_N > 0$ and negative if $b_N < 0$. This means further that (see (3.5.a)), for $u < \bar{t}_1$, $w_l(u)$ increases if N is odd, i.e., $-d_l > 0$, and decreases if N is even, i.e., $-d_l < 0$. This completes the proof of the theorem.

Theorem A allows us to compare the graphs of the error functions $w_e(u)$, $w_l(u)$ with the corresponding error function $T_{2N+1}(u)$ for the case of polynomial interpolation. The differences are

$$\operatorname{sgn} w_e'(0) = -1, \qquad \operatorname{sgn} w_e(1) = (-1)^{N+1};$$

$$\operatorname{sgn} w_l'(0) = (-1)^{N+1}, \qquad \operatorname{sgn} w_l(1) = -1;$$

$$\operatorname{sgn} T_{2N+1}'(0) = (-1)^N, \qquad \operatorname{sgn} T_{2N+1}(1) = 1.$$

We then see from (2.11) and (2.15) that the signs of $h_e'(0)$, $h_l'(0)$, and $T_{2N+1}'(0)$ are identical, just like the signs of $h_e(1)$, $h_l(1)$, and $T_{2N+1}(1)$.

4. Behavior of Chebyshev Quantities for Small Values of α

Theorem B.

$$\lim_{\alpha\to 0} e_i = \lim_{\alpha\to 0} t_i = p_i\,, \qquad \lim_{\alpha\to 0} \bar{e}_i = \lim_{\alpha\to 0} \bar{t}_i = \bar{p}_i\,, \qquad i = 1, 2,..., N,$$

where p_i and $\bar{p}_i$ are given by (3.7); $\lim_{\alpha\to 0} d_e = \lim_{\alpha\to 0} d_t = 0$.

Proof. We first consider the exponential case. We expand the function $w_e(u)$ in a Maclaurin series, to give

$$w_e(u) = \alpha u \sum_{k=1}^{N} k a_k - u + \frac{(\alpha u)^3}{3!} \sum_{k=1}^{N} k^3 a_k + \cdots$$
$$+ \frac{(\alpha u)^{2N+1}}{(2N+1)!} \sum_{k=1}^{N} k^{2N+1} a_k + \cdots. \tag{4.1}$$

When putting formally

$$\alpha \sum_{k=1}^{N} k^{2p+1} a_k = \sum_{m=0}^{\infty} \lambda_{2p+1,2m} \alpha^{2m}, \qquad p = 0, 1, 2,... \tag{4.2}$$

(valid for small values of α) and

$$g_{2p+1}(u) = \sum_{m=0}^{p} \frac{\lambda_{2m+1,2p-2m}}{(2m+1)!} u^{2m+1}, \qquad p = 1, 2,..., \tag{4.3}$$

we can rewrite (4.1) as

$$w_e(u) = (\lambda_{1,0} - 1)u + \sum_{p=1}^{\infty} g_{2p+1}(u)\, \alpha^{2p}. \tag{4.4}$$

We also have for the derivative

$$w_e'(u) = \lambda_{1,0} - 1 + \sum_{p=1}^{\infty} g'_{2p+1}(u)\, \alpha^{2p}. \tag{4.5}$$

Now we introduce the following formal series for extremal points and zeros, valid for small values of α:

$$\bar{e}_i = \sum_{m=0}^{\infty} \bar{C}_{e,i,2m} \alpha^{2m}; \qquad e_i = \sum_{m=0}^{\infty} C_{e,i,2m} \alpha^{2m}. \tag{4.6}$$

Then

$$g'_{2p+1}(\bar e_i) = g'_{2p+1}(\bar C_{e,i,0}) + \frac{1}{1!}\left(\sum_{m=1}^{\infty} \bar C_{e,i,2m}\alpha^{2m}\right) g''_{2p+1}(\bar C_{e,i,0}) + \cdots$$
$$+ \frac{1}{(2p-1)!}\left(\sum_{m=1}^{\infty} \bar C_{e,i,2m}\alpha^{2m}\right)^{2p-1} g^{(2p)}_{2p+1}(\bar C_{e,i,0}). \tag{4.7}$$

Putting this into (4.5) yields

$$w_e'(\bar e_i) = 0 = \lambda_{1,0} - 1 + \sum_{p=1}^{\infty} G'_{2p+1}(\bar C_{e,i,0})\,\alpha^{2p}, \tag{4.8}$$

where G'_{2p+1} is a polynomial of degree $2p$. For the lower degree polynomials G_3', G_5' we have

$$G_3'(\bar C_{e,i,0}) = g_3'(\bar C_{e,i,0}); \qquad G_5'(\bar C_{e,i,0}) = g_5'(\bar C_{e,i,0}) + \bar C_{e,i,2}\, g_3''(\bar C_{e,i,0}).$$

Since, from (4.8), $G'_{2p+1}(\bar C_{e,i,0}) = 0$, $i = 1, 2,\dots, N$, all p, and since the polynomials $g'_{2p+1}(u)$ are even and, hence, have at most p positive zeros, we can conclude that

$$\lambda_{1,0} = 1; \qquad g_3(u) = g_5(u) = \cdots = g_{2N-1}(u) = 0. \tag{4.9}$$

It follows further from (4.4), (4.7), and (4.9) that

$$w_e(\bar e_i) = (-1)^i d_e = \sum_{p=N}^{\infty} G_{2p+1}(\bar C_{e,i,0})\,\alpha^{2p}, \qquad i = 1, 2,\dots, N, \tag{4.10}$$

where G_{2p+1} is a polynomial of degree $2p+1$. In particular we have

$$G_{2N+1}(\bar C_{e,i,0}) = g_{2N+1}(\bar C_{e,i,0}); \tag{4.11.a}$$

$$G_{2N+3}(\bar C_{e,i,0}) = g_{2N+3}(\bar C_{e,i,0}) + \bar C_{e,i,2}\, g'_{2N+1}(\bar C_{e,i,0}). \tag{4.11.b}$$

From (4.10) we see that $d_e = O(\alpha^{2N})$, and we can put formally

$$d_e = \sum_{p=N}^{\infty} q_{2p}\alpha^{2p}. \tag{4.12}$$

Since (Theorem A) $d_e > 0$, we know that $q_{2N} > 0$.

It follows from (4.4), (4.9), and (4.12) that the extremal points $\pm\bar e_i$, ± 1, in the limit $\alpha = 0$, are zeros of the polynomials $q_{2N}^2 - g_{2N+1}^2(u)$ and $(1-u^2)\,g'^2_{2N+1}(u)$. The polynomial $g_{2N+1}(u)$ thus satisfies the differential equation

$$(1-u^2)\,g'^2_{2N+1}(u) = (2N+1)^2[q_{2N}^2 - g^2_{2N+1}(u)],$$

the solution of which is known [7] to be proportional to the Chebyshev polynomial of the first kind $T_{2N+1}(u)$. More precisely we have

$$g_{2N+1}(u) = (-1)^{N+1} q_{2N} T_{2N+1}(u), \tag{4.13}$$

since $\operatorname{sgn} w_e(1) = (-1)^{N+1}$ and $T_{2N+1}(1) = 1$. Now the theorem (for the exponential case) follows at once from (4.6), (4.11.a), and (4.13).

We can repeat the same calculations for the trigonometric case. Starting from the Maclaurin series for $w_t(u)$, and putting

$$\alpha \sum_{k=1}^{N} k^{2p+1} b_k = \sum_{m=0}^{\infty} \mu_{2p+1,2m} \alpha^{2m}, \qquad p = 0, 1, 2, \ldots,$$

$$h_{2p+1}(u) = \sum_{m=0}^{p} (-1)^m \frac{\mu_{2m+1,2p-2m}}{(2m-1)!} u^{2m+1}, \qquad p = 1, 2, \ldots,$$

$$\bar{t}_i = \sum_{m=0}^{\infty} \bar{C}_{t,i,2m} \alpha^{2m}, \qquad t_i = \sum_{m=0}^{\infty} C_{t,i,2m} \alpha^{2m},$$

it is easy to show that $d_t = O(\alpha^{2N})$; hence, we put formally

$$d_t = \sum_{p=N}^{\infty} r_{2p} \alpha^{2p}, \tag{4.14}$$

where, by Theorem A, $\operatorname{sgn} r_{2N} = (-1)^N$. Further, it is readily seen that

$$h_{2N+1}(u) = (-1)^{N+1} r_{2N} T_{2N+1}(u), \tag{4.15}$$

which proves the theorem for the trigonometric case.

We now wish to calculate $\lim_{\alpha\to 0} \epsilon_e$ and $\lim_{\alpha \to 0} \epsilon_t$. Therefore, we need to know q_{2N} and r_{2N}. The calculation is based on the following lemma.

LEMMA a. *We have*

$$\begin{aligned} &\operatorname{sgn} \lambda_{2N+2p+1,0} = (-1)^{N+1}, \qquad \operatorname{sgn} \lambda_{2N+2p+1,2} = (-1)^N, \\ &\operatorname{sgn} \mu_{2N+2p+1,0} = \operatorname{sgn} \mu_{2N+2p+1,2} = (-1)^{N+1}, \end{aligned} \qquad p = 0, 1, 2, \ldots;$$

further we have

$$\lambda_{2N+1,0} = \mu_{2N+1,0} = (-1)^{N-1}(N!)^2; \tag{4.16.a}$$

$$\lambda_{2N+3,0} = \mu_{2N+3,0} = (-1)^{N-1}(N!)^2 \left(\sum_{k=1}^{N} k^2\right); \tag{4.16.b}$$

$$\lambda_{2N+1,2} = \left(\sum_{k=1}^{N} k^2\right) \lambda_{2N-1,2}\,; \qquad \mu_{2N+1,2} = \left(\sum_{k=1}^{N} k^2\right) \mu_{2N-1,2}\,. \tag{4.16.c}$$

Proof. We consider the set of infinitely many linear equations (4.2) with the unknowns $\alpha k a_k = k a_k^*$, $k = 1, 2, \ldots, N$. We define $V_N^{(N-1)}$ and $V_N^{(N+p)}$ to be

$$V_N^{(N-1)} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1^2 & 2^2 & \cdots & N^2 \\ \cdot & \cdot & \cdots & \cdot \\ 1^{2N-4} & 2^{2N-4} & \cdots & N^{2N-4} \\ 1^{2N-2} & 2^{2N-2} & \cdots & N^{2N-2} \end{vmatrix}; \qquad V_N^{(N+p)} = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ 1^2 & 2^2 & \cdots & N^2 \\ \cdot & \cdot & \cdots & \cdot \\ 1^{2N-4} & 2^{2N-4} & \cdots & N^{2N-4} \\ 1^{2N+2p} & 2^{2N+2p} & \cdots & N^{2N+2p} \end{vmatrix}.$$

These are determinants of Vandermonde type and, hence, have a positive value. It is not difficult to show the relation

$$V_N^{(N)} = \left(\sum_{k=1}^{N} k^2\right) V_N^{(N-1)}. \tag{4.17}$$

From the first N equations (4.2) and from (4.9) we get

$$Na_N^* = (-1)^{N-1}[(N-1)!]^2 \frac{V_{N-1}^{(N-2)}}{V_N^{(N-1)}} + \frac{V_{N-1}^{(N-2)}}{V_N^{(N-1)}} \lambda_{2N-1,2}\alpha^2 + 0(\alpha^4). \tag{4.18}$$

The second, thirth,..., Nth, and $(N+p+1)$th equations (4.2) together give

$$Na_N^* = \frac{1}{N^2}\Bigg\{\frac{V_{N-1}^{(N-2)}}{V_N^{(N+p-1)}} \lambda_{2N+2p+1,0}$$

$$+ \left[\frac{V_{N-1}^{(N-2)}}{V_N^{(N+p-1)}} \lambda_{2N+2p+1,2} - \frac{V_{N-1}^{(N+p-1)}}{V_N^{(N+p-1)}} \lambda_{2N-1,2}\right] \alpha^2\Bigg\} + 0(\alpha^4). \tag{4.19}$$

From (4.18) and (4.19) we infer

$$\lambda_{2N+2p+1,0} = (-1)^{N-1}(N!)^2 \frac{V_N^{(N+p-1)}}{V_N^{(N-1)}};$$

$$\lambda_{2N+2p+1,2} = \frac{N^2 V_N^{(N+p-1)}}{V_{N-1}^{(N-2)}} \left[\frac{V_{N-1}^{(N-2)}}{V_N^{(N-1)}} + \frac{V_{N-1}^{(N+p-1)}}{N^2 V_N^{(N+p-1)}}\right] \lambda_{2N-1,2},$$

where $\operatorname{sgn} \lambda_{2N-1,2} = (-1)^N$. From this and Eq. (4.17) the lemma follows for the λ-quantities. By repeating the same calculations, the lemma appears also to be true for the μ-quantities (however, observe that $\operatorname{sgn} \mu_{2N-1,2} = (-1)^{N-1}$).

THEOREM C.

$$\lim_{\alpha\to 0} \epsilon_r = \lim_{\alpha\to 0} \epsilon_t = (-1)^{N+1}\epsilon_p, \text{ where } \epsilon_p \text{ is given by (3.8).}$$

Proof. Equating the coefficients of u^{2N+1} in both sides of Eqs. (4.13) and (4.15) gives

$$\frac{\lambda_{2N+1,0}}{(2N+1)!} = (-1)^{N+1} q_{2N} 2^{2N} = \frac{\mu_{2N+1,0}}{(2N+1)!} = -r_{2N} 2^{2N}.$$

On account of (4.16) we obtain

$$q_{2N} = (-1)^N r_{2N} = \frac{(N!)^2}{2^{2N}(2N+1)!}$$

(and, hence, $h_{2N+1}(u) = (-1)^N g_{2N+1}(u)$). The theorem then follows from (3.6), (4.12), and (4.14).

In principle we can calculate all quantities $\lambda_{2m+1,2p}$, q_{2p}, $\bar{C}_{e,i,2m}$, and $C_{e,i,2m}$ from the infinite set of equations (4.2), from (4.10) and (4.12), and using the fact that we already know the coefficients of $g_{2N+1}(u)$. The same is true for the quantities related to the trigonometric case. In particular, knowledge of q_{2N+2} and r_{2N+2}, $\bar{C}_{e,i,2}$ and $\bar{C}_{t,i,2}$, $C_{e,i,2}$ and $C_{t,i,2}$ informs us about the increasing or decreasing behavior (for small values of α) of the quantities ϵ_e, $\bar{e}_i$ and $\bar{t}_i$, e_i and t_i, respectively. Results are established in the next two theorems.

THEOREM D. *We have*

$$\bar{C}_{e,i,2} = -\frac{g'_{2N+3}(\bar{p}_i)}{g''_{2N+1}(\bar{p}_i)}; \qquad C_{e,i,2} = -\frac{g_{2N+3}(p_i)}{g'_{2N+1}(p_i)}; \tag{4.20}$$

$$\bar{C}_{t,i,2} = -\bar{C}_{e,i,2}; \qquad C_{t,i,2} = -C_{e,i,2}. \tag{4.21}$$

Proof. From (4.8), (4.9), and Theorem B we get

$$G'_{2N+3}(\bar{p}_i) = 0 = g'_{2N+3}(\bar{p}_i) + \bar{C}_{e,i,2} g''_{2N+1}(\bar{p}_i).$$

Since $g''_{2N+1}(\bar{p}_i) \neq 0$ (see (4.13)), the first equation (4.20) is proved. The second equation (4.20) can be derived in an analogous manner, starting from the equation $w_e(e_i) = 0$, and using (4.11).

Now it follows from the calculation procedure of the λ- and μ-quantities that $|\lambda_{i,j}| = |\mu_{i,j}|$, all i, j. Hence, we can infer from Lemma a and the definition of $h_{2N+3}(u)$ that $g_{2N+3}(u) = (-1)^{N+1} h_{2N+3}(u)$. Equations (4.21) are immediate consequences of this.

THEOREM E.

$$q_{2N+2} < 0; \qquad r_{2N+2} = (-1)^N q_{2N+2}, \quad \text{i.e.,} \quad \operatorname{sgn} r_{2N+2} = \operatorname{sgn} r_{2N}.$$

Proof. Consider first the exponential case. It follows from (4.10), (4.11.b), (4.12), and (4.13) that

$$G_{2N+3}(\bar{p}_i) - g_{2N+3}(\bar{p}_i) = (-1)^i q_{2N+2}, \qquad i = 1, 2, \dots, N+1,$$

where $\bar{p}_{N+1} = 1$. On account of the definition (4.3) of g_{2N+3} we then have

$$\begin{aligned} &\lambda_{1,2N+2}\,\bar{p}_i + \frac{\lambda_{3,2N}}{3!}\,\bar{p}_i^{3} + \cdots + \frac{\lambda_{2N-1,4}}{(2N-1)!}\,\bar{p}_i^{2N-1} + (-1)^{i-1} q_{2N+2} \\ &\quad = -\frac{\lambda_{2N+1,2}}{(2N+1)!}\,\bar{p}_i^{2N+1} - \frac{\lambda_{2N+3,0}}{(2N+3)!}\,\bar{p}_i^{2N+3}, \qquad i = 1, 2, \dots, N+1. \end{aligned} \tag{4.22}$$

The quantities in the right members of these equations are known from Lemma a. When considering $\lambda_{2m+1,2N-2m+2}/(2m+1)!$, $m = 0, 1, \dots, N-1$, and q_{2N+2} as unknowns, the determinant D of the system (4.22) is a sum of determinants of Vandermonde type; it is readily seen that

$$\operatorname{sgn} D = (-1)^N. \tag{4.23}$$

Solving the system (4.22) for q_{2N+2} gives

$$q_{2N+2} = -\frac{1}{D}\left\{\frac{\lambda_{2N+1,2}}{(2N+1)!}\begin{vmatrix} \bar{p}_1 & \bar{p}_1^{3} & \cdots & \bar{p}_1^{2N-1} & \bar{p}_1^{2N+1} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \bar{p}_N & \bar{p}_N^{3} & \cdots & \bar{p}_N^{2N-1} & \bar{p}_N^{2N+1} \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix} + \frac{\lambda_{2N+3,0}}{(2N+3)!}\begin{vmatrix} \bar{p}_1 & \bar{p}_1^{3} & \cdots & \bar{p}_1^{2N-1} & \bar{p}_1^{2N+3} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ \bar{p}_N & \bar{p}_N^{3} & \cdots & \bar{p}_N^{2N-1} & \bar{p}_N^{2N+3} \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}\right\}.$$

By virtue of a relation similar to (4.17) we have

$$\begin{vmatrix} 1 & \bar{p}_1^{2} & \cdots & \bar{p}_1^{2N-2} & \bar{p}_1^{2N+2} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \bar{p}_N^{2} & \cdots & \bar{p}_N^{2N-2} & \bar{p}_N^{2N+2} \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix} = \sum_{k=1}^{N+1} \bar{p}_k^{2} \begin{vmatrix} 1 & \bar{p}_1^{2} & \cdots & \bar{p}_1^{2N-2} & \bar{p}_1^{2N} \\ \cdot & \cdot & \cdots & \cdot & \cdot \\ 1 & \bar{p}_N^{2} & \cdots & \bar{p}_N^{2N-2} & \bar{p}_N^{2N} \\ 1 & 1 & \cdots & 1 & 1 \end{vmatrix}.$$

Denoting the determinant on the left side by P, we obtain for q_{2N+2}

$$q_{2N+2} = -\left(\prod_{i=1}^{N} \bar{p}_i\right)\frac{P}{D}\left[\frac{\lambda_{2N+1,2}}{(2N+1)!} + \left(\sum_{k=1}^{N+1} \bar{p}_k^{2}\right)\frac{\lambda_{2N+3,0}}{(2N+3)!}\right]. \tag{4.24}$$

In order to calculate the expression between brackets we must calculate first $\lambda_{2N+1,2}$ or, by Eq. (4.16.b), $\lambda_{2N-1,2}$. We have [7]

$$T_{2N+1}(u) = \sum_{k=0}^{N} c_k u^{2N-2k+1}, \quad c_k = (-1)^k \, 2^{2N-2k} \frac{2N+1}{2N-k+1} \binom{2N-k+1}{k}. \tag{4.25}$$

Hence, from (4.13), (4.3), and the expression for q_{2N} we obtain, putting $k = 1$,

$$\frac{\lambda_{2N-1,2}}{(2N-1)!} = (-1)^N \frac{(N!)^2}{2^2(2N)!}.$$

Consequently, by (4.16.b),

$$\frac{\lambda_{2N+1,2}}{(2N-1)!} = (-1)^N \left(\sum_{k=1}^{N} k^2\right) \frac{(N!)^2}{(2N+1)!\, 2^2 \cdot 2N}. \tag{4.26}$$

We also infer from (4.25) that

$$\sum_{i=1}^{N} \bar{p}_i^{\,2} = -\frac{(2N-1)\, c_1}{(2N+1)\, c_0} = \frac{2N-1}{2^2},$$

and, hence, that

$$\sum_{i=1}^{N+1} \bar{p}_i^{\,2} = \frac{2N+3}{2^2}. \tag{4.27}$$

Putting (4.26) and (4.27) into (4.24) yields

$$q_{2N+2} = (-1)^{N+1} \left(\prod_{i=1}^{N} \bar{p}_i\right)\left(\sum_{k=1}^{N} k^2\right) \frac{P}{D} \frac{(N!)^2}{2^2 \cdot 2N(2N+2)!}.$$

It then follows from (4.23) that $q_{2N+2} < 0$.

The theorem is now easily proved for the trigonometric case, too. Instead of (4.22) we now have a system with unknowns $(-1)^m \mu_{2m+1,2N-2m+2}/(2m+1)!$, $m = 0, 1,\ldots, N-1$, and r_{2N+2}, with the same determinant D. The right members are

$$(-1)^{N+1} \frac{\mu_{2N+1,2}}{(2N+1)!} \bar{p}_i^{2N+1} + (-1)^{N+2} \frac{\mu_{2N+3,0}}{(2N+3)!} \bar{p}_i^{2N+3}$$
$$= (-1)^N \left[\frac{\lambda_{2N+1,2}}{(2N+1)!} \bar{p}_i^{2N+1} + \frac{\lambda_{2N+3,0}}{(2N+3)!} \bar{p}_i^{2N+3}\right]$$

(since $\mu_{2N+3,0} = \lambda_{2N+3,0}$ and $\mu_{2N+1,2} = -\lambda_{2N+1,2}$). Hence, it appears from (4.24) that $r_{2N+2} = (-1)^{N+1} q_{2N+2}$.

5. Behavior of Chebyshev Quantities for Large Values of α

We can summarize the results of this section in the following theorem.

THEOREM F.

$$|d_t|_{\alpha=\pi} = 1; \qquad |\epsilon_t|_{\alpha=\pi} > \epsilon_p ;$$

$$\lim_{\alpha\to\infty} d_e = \lim_{\alpha\to\infty} \bar{e}_i = \lim_{\alpha\to\infty} e_i = 1, \qquad i = 1, 2,\dots, N; \quad \lim_{\alpha\to\infty} \epsilon_e = 0.$$

Proof. We consider first the trigonometric case, being the simplest one. If $\alpha = \pi$, it follows at once from (3.5.a) that $d_t|_{\alpha=\pi} = (-1)^N$. A more detailed calculation shows that, for α close to π,

$$d_t = (-1)^N \left\{1 - (\pi - \alpha) \sum_{k=1}^{N} (-1)^{k+1} kB_k + 0[(\pi - \alpha)^2]\right\},$$

where $B_k = b_k|_{\alpha=\pi}$. By virtue of Theorem A the expression $\sum_{k=1}^{N} (-1)^{k-1} B_k$ is positive; hence, $|d_t|$ increases towards the limiting value 1. For ϵ_t we have $\epsilon_t|_{\alpha=\pi} = (-1)^{N-1}/(\pi^N N!)^2$. It is not difficult to show that $(\pi^N N!)^2 < 2^{2N}(2N+1)!$; hence, $|\epsilon_t|_{\alpha=\pi} > \epsilon_p$.

Let us now turn to the exponential case. We first consider Eq. (3.4.b), written in exponential form, and express the coefficients ka_k as functions of the roots $\exp(\pm\bar{e}_i{}^*)$ by means of the elementary symmetrical functions of the roots (see [6]). These can be written compactly as follows:

$$(N-2j)\, a_{N-2j} = Na_N \sum_{l=0}^{j} \left\{\binom{N-2l}{j-l} 2^{2l} \sum_{i_1<i_2<\cdots<i_{2l}}^{1,N} \left(\prod_{m=1}^{2l} \cosh \bar{e}_{i_m}^*\right)\right\}, \qquad j = 1, 2,\dots, [(N-1)/2],$$

$$(N-2j-1)\, a_{N-2j-1} = -Na_N \sum_{l=0}^{j} \left\{\binom{N-2l-1}{j-l} 2^{2l+1} \sum_{i_1<i_2<\cdots<i_{2l+1}}^{1,N} \left(\prod_{m=1}^{2l+1} \cosh \bar{e}_{i_m}^*\right)\right\}, \qquad j = 0, 1,\dots, [(N-2)/2],$$

$$-\frac{2}{\alpha} = \begin{cases} Na_N \displaystyle\sum_{l=0}^{N/2} \left\{\binom{N-2l}{N/2-l} 2^{2l} \sum_{i_1<i_2<\cdots<i_{2l}}^{1,N} \left(\prod_{m=1}^{2l} \cosh \bar{e}_{i_m}^*\right)\right\}, & N \text{ even.} \\ -Na_N \displaystyle\sum_{l=0}^{(N-1)/2} \left\{\binom{N-2l-1}{(N-1)/2-l} 2^{2l+1} \sum_{i_1<i_2<\cdots<i_{2l+1}}^{1,N} \left(\prod_{m=0}^{2l+1} \cosh \bar{e}_{i_m}^*\right)\right\}, & N \text{ odd.} \end{cases}$$

From these formulas we can detect the behavior of the a_k for large values of α. Replacing $\cosh \bar{e}^*_{i_m}$ by $\exp(\bar{e}^*_{i_m})/2$ we obtain

$$ka_k^* \sim (-1)^{k+1}\, 2 \sum_{i_1<i_2<\cdots<i_k}^{1,N} \exp\left(-\sum_{m=1}^{k} \bar{e}^*_{i_m}\right), \qquad k = 1, 2,\ldots, N. \quad (5.1)$$

Doing the same replacements in the equation $w_e'(\bar{e}_j) = 0$ we get

$$\sum_{k=1}^{N} (ka_k^*/2)\, e^{k\bar{e}_j^*} \sim 1, \qquad j = 1, 2,\ldots, N.$$

or, on account of (5.1),

$$\sum_{k=1}^{N} \left\{(-1)^{k+1} \sum_{i_1<i_2<\cdots<i_k}^{1,N} \exp\left[\sum_{m=1}^{k} (\bar{e}_j^* - \bar{e}^*_{i_m})\right]\right\} \sim 1, \qquad j = 1, 2,\ldots, N. \quad (5.2)$$

If $j = 1$, Eq. (5.2) is satisfied for $\alpha = \infty$ since the last term of the left member is equal to 1, while all other terms are 0, their exponents being negative. If $j \geqslant 2$, then the left member of (5.2) will have a finite value only if

$$\lim_{\alpha\to\infty} \bar{e}_1 = \lim_{\alpha\to\infty} \bar{e}_2 = \cdots = \lim_{\alpha\to\infty} \bar{e}_N = L, \qquad L \leqslant 1. \quad (5.3)$$

For the equalities $w_e(\bar{e}_j) = (-1)^j d_e$, $j = 1, 2,\ldots, N$ we have

$$\sum_{k=1}^{N} \frac{a_k}{2} e^{k\bar{e}_j^*} - \bar{e}_j \sim (-1)^j d_e, \qquad j = 1, 2,\ldots, N,$$

or, on account of (5.1),

$$\sum_{k=1}^{N} \frac{(-1)^{k+1}}{k} \left\{\sum_{i_1<i_2<\cdots<i_k}^{1,N} \exp\left[\sum_{m=1}^{k} (\bar{e}_j^* - \bar{e}^*_{i_m})\right]\right\} \sim \bar{e}_j^* + (-1)^j d_e^*,$$
$$j = 1, 2,\ldots, N. \quad (5.4)$$

Again, for $j = 1$, the limiting value of the left member is 1; hence, $\lim_{\alpha\to\infty} d_e = \lim_{\alpha\to\infty} \bar{e}_1 = L$. If we apply (5.1) to the equation $w_e(1) = (-1)^{N+1} d_e$, we obtain

$$\alpha^{-1} \sum_{k=1}^{N} \frac{(-1)^{k+1}}{k} \left\{\sum_{i_1<i_2<\cdots<i_k}^{1,N} \exp\left[\sum_{m=1}^{k} (\alpha - \bar{e}^*_{i_m})\right]\right\} \sim 1 + (-1)^{N+1} d_e. \quad (5.5)$$

Since the right member remains finite, the same is true for the left member; there the leading term is

$$(\alpha N)^{-1}(-1)^{N+1}\exp\left[\alpha\sum_{i=1}^{N}(1-\bar{e}_i)\right].$$

For this to be finite we must have $\lim_{\alpha\to\infty}\bar{e}_i = 1$, i.e., $L = 1$. Since d_e remains finite, $\lim_{\alpha\to\infty}\epsilon_e = 0$. This completes the proof of the theorem.

From (5.5) we can deduce further that

$$\bar{e}_i \sim 1 - \bar{E}_i \log\alpha/\alpha, \qquad i = 1, 2,\ldots, N,$$

where the constants $\bar{E}_i$ are such that $\bar{E}_i - \bar{E}_{i+1} > 0$. For the zeros e_i similar asymptotic expansions hold. Taking $j = 1$ in (5.4) we obtain that $\lim_{\alpha\to\infty}\alpha(\bar{e}_1 - d_e) = 1$; hence,

$$d_e \sim 1 - \bar{E}_1 \log\alpha/\alpha - 1/\alpha.$$

This shows that $|\epsilon_e|$ decreases towards the limiting value 0.

6. Global Behavior of Chebyshev Quantities

In this section primes denote differentiation with respect to α. We first demonstrate a monotonicity property of Chebyshev quantities related to the interval $[-\alpha, \alpha]$.

Theorem G. *The quantities* d_e^*, $|d_i^*|$, $\bar{e}_i^*$, $\bar{t}_i^*$, e_i^*, t_i^*, $i = 1, 2,\ldots, N$ *are monotonically increasing functions of* α.

Proof. We give the proof for the exponential case, the other case being completely similar. Consider two values α_1 and α_2 of α, $\alpha_1 < \alpha_2$. The function

$$\sum_{k=1}^{N}[Ka_k^*(\alpha_1) - a_k^*(\alpha_2)]\sinh ky + (K - 1)y, \tag{6.1}$$

where $K = d_e^*(\alpha_2)/d_e^*(\alpha_1)$, is defined for all values of α and has at least N positive zeros, one in each interval $(\bar{e}_i^*(\alpha_1), \bar{e}_{i+1}^*(\alpha_1)]$, $i = 1, 2,\ldots, N$, with $\bar{e}_{N+1}^* = \alpha$. If K should be equal to 1, a linear combination of the N functions $\sinh ky$, $k = 1, 2,\ldots, N$, should have at least N positive zeros, which is impossible since the system of these functions satisfies the Haar condition on any interval $(0, \alpha]$. Since $d_e^*|_{\alpha=0} = 0$ (Theorem B), the theorem is proved for d_e^*.

If $\bar{e}_1^*(\alpha_1) > \bar{e}_1^*(\alpha_2)$, then there should lie a zero of the function (6.1) in the interval $(0, \bar{e}_1^*(\alpha_1)]$, too. This is impossible since then this function should have at least $2N + 3$ zeros, while it is clear that it can have at most $2N + 1$ zeros. Suppose now there is a j, $j \geq 1$, such that $\bar{e}_j^*(\alpha_2) > \bar{e}_j^*(\alpha_1)$ and $\bar{e}_{j+1}^*(\alpha_2) < \bar{e}_{j+1}^*(\alpha_1)$. The interval $(\bar{e}_j^*(\alpha_1), \bar{e}_{j+1}^*(\alpha_1)]$ then contains at least three zeros, which brings the total number of positive zeros of (6.1) up to at least $N + 2$, again impossible. This proves the theorem for the extremal points $\bar{e}_i^*$ and, by using the same reasoning, also for the zeros e_i^*.

We next show some monotonicity properties of the a_k^* and b_k^* coefficients.

THEOREM H. *The coefficients* $|a_k^*|$ *are monotonically decreasing functions of* α, *i.e.*, $\operatorname{sgn} a_k^{*\prime} = (-1)^k$. *The coefficients* $|b_k^*|$ *are monotonically increasing functions of* α, *i.e.*, $\operatorname{sgn} b_k^{*\prime} = (-1)^{k+1}$.

Proof. It follows from (3.4), (3.5) that

$$\sum_{k=1}^{N} a_k^* \sinh k\bar{e}_i^* - \bar{e}_i^* = (-1)^i d_e^*, \qquad i = 1, 2,..., N + 1, \tag{6.2.a}$$

$$\sum_{k=1}^{N} b_k^* \sin k\bar{t}_i^* - \bar{t}_i^* = (-1)^i d_t^*, \qquad i = 1, 2,..., N + 1, \tag{6.2.b}$$

where $\bar{e}_{N+1}^* = \bar{t}_{N+1}^* = \alpha$. Differentiation of (6.2) with respect to α yields

$$\begin{gathered}\sum_{k=1}^{N} a_k^{*\prime} \sinh k\bar{e}_i^* = (-1)^i d_e^{*\prime}, \qquad i = 1, 2,..., N,\\ \sum_{k=1}^{N} a_k^{*\prime} \sinh k\alpha + \sum_{k=1}^{N} ka_k^* \cosh k\alpha - 1 = (-1)^{N+1} d_e^{*\prime},\end{gathered} \tag{6.3.a}$$

and

$$\begin{gathered}\sum_{k=1}^{N} b_k^{*\prime} \sin k\bar{t}_i^* = (-1)^i d_t^{*\prime}, \qquad i = 1, 2,..., N,\\ \sum_{k=1}^{N} b_k^{*\prime} \sin k\alpha + \sum_{k=1}^{N} kb_k^* \cos k\alpha - 1 = (-1)^{N+1} d_t^{*\prime}.\end{gathered} \tag{6.3.b}$$

It appears from these equations that the functions

$$f_e(y) = \sum_{k=1}^{N} a_k^{*\prime} \sinh ky, \qquad f_t(y) = \sum_{k=1}^{N} b_k^{*\prime} \sin ky$$

have the largest possible number of positive zeros, i.e., $N - 1$. Since $d_e^{*\prime} > 0$ (Theorem G), it follows that $\operatorname{sgn} a_N^{*\prime} = (-1)^N$. Furthermore, by applying the same reasoning as in the proof of Theorem A, the $a_k^{*\prime}$ must differ from zero and have alternating signs. Hence, $\operatorname{sgn} a_k^{*\prime} = (-1)^k$. For the trigonometric case we have (Theorem G) $\operatorname{sgn} d_t^{*\prime} = (-1)^N$, and, hence, $\operatorname{sgn} f_t(\bar{t}_1^*) = (-1)^{N+1}$; this means that $\operatorname{sgn} b_N^{*\prime} = (-1)^{N+1}$, which further implies that $\operatorname{sgn} b_k^{*\prime} = (-1)^{k+1}$.

THEOREM I. *The quantities* $|\sum_{k=1}^{N} ka_k^* - 1|$, $|\sum_{k=1}^{N} ka_k^* \cosh k\alpha - 1|$, $|\sum_{k=1}^{N} kb_k^* - 1|$ *and* $|\sum_{k=1}^{N} kb_k^* \cos k\alpha - 1|$ *are monotonically increasing functions of* α, *i.e.*, $\sum_{k=1}^{N} ka_k^{*\prime} < 0$, $\operatorname{sgn}(\sum_{k=1}^{N} ka_k^* \cosh k\alpha - 1)' = (-1)^{N+1}$, $\operatorname{sgn} \sum_{k=1}^{N} kb_k^{*\prime} = (-1)^{N+1}$, *and* $(\sum_{k=1}^{N} kb_k^* \cos k\alpha - 1)' < 0$.

Proof. Since from (6.3), $\operatorname{sgn} f_e(\bar{e}_1^*) = -1$ and $\operatorname{sgn} f_t(\bar{t}_1^*) = (-1)^{N+1}$, it immediately follows that $\sum_{k=1}^{N} ka_k^{*\prime} < 0$ and $\operatorname{sgn} \sum_{k=1}^{N} kb_k^{*\prime} = (-1)^{N+1}$. Consider now the functions

$$\mathscr{F}_e(y) = f_e(y) + \sum_{k=1}^{N} ka_k^* \cosh k - 1, \tag{6.4.a}$$

$$d\mathscr{F}_e(y)/dy = \sum_{k=1}^{N} ka_k^{*\prime} \cosh ky + \sum_{k=1}^{N} k^2 a_k^* \sinh ky. \tag{6.4.b}$$

The first can have at most $2N$ real zeros, the second at most $2N - 1$. We have

$$\mathscr{F}(\bar{e}_i^*) = (-1)^i d_e^{*\prime}, \qquad i = 1, 2, \ldots, N + 1,$$

with $\bar{e}_{N+1}^* = \alpha$. Since $d_e^{*\prime} > 0$ and $\operatorname{sgn}(\sum_{k=1}^{N} ka_k^* \cosh k\alpha - 1) = (-1)^{N+1}$, $\mathscr{F}_e(y)$ has at least N positive zeros and, since $\mathscr{F}_e(\bar{e}_i^*) = \mathscr{F}_e(-\bar{e}_i^*)$ and $\sum_{k=1}^{N} ka_k^* - 1 < 0$, also at least N negative zeros. Hence, it is impossible that $d\mathscr{F}_e(\alpha)/dy = (\sum_{k=1}^{N} ka_k^* \cosh k\alpha - 1)' = 0$, and $\sum_{k=1}^{N} ka_k^* \cosh k\alpha - 1$ is a monotonic function of α. It now follows from (4.2) and (4.9) that, for small values of α, this function is $O(\alpha^{2N})$ and, hence, has the same sign as its derivative. This is consequently true for all values of α. Exactly the same reasoning applies for the quantity $\sum_{k=1}^{N} kb_k^* \cos k\alpha - 1$.

We now proceed with monotonicity properties of Chebyshev quantities related to the interval $[-1, 1]$. These are based on two lemmas.

LEMMA b. *If* α_1 *and* α_2 *are two values of* α *satisfying the condition*

$$[(N - 1)/N]\,\alpha_2 < \alpha_1 < \alpha_2, \tag{6.5}$$

then the functions

$$F_e(u) = \sum_{k=1}^{N} ka_k^*(\alpha_1)\cosh\alpha_1 ku - \sum_{k=1}^{N} ka_k^*(\alpha_2)\cosh\alpha_2 ku, \tag{6.6.a}$$

$$F_t(u) = \sum_{k=1}^{N} kb_k^*(\alpha_1)\cos\alpha_1 ku - \sum_{k=1}^{N} kb_k^*(\alpha_2)\cos\alpha_2 ku, \tag{6.6.b}$$

can have at most $2N$ *real zeros (of which* N *positive).*

Proof. Suppose N even, and assume that $F_e(u)$ has more than $2N$ real zeros. When rewriting $F_e(u)$ as

$$F_e(u) = \tfrac{1}{2} e^{\alpha_1 u} F_1(u),$$

where

$$F_1(u) = a_1^*(\alpha_1) + a_1^*(\alpha_1)\, e^{-2\alpha_1 u} + \sum_{k=2}^{N} ka_k^*(\alpha_2)[e^{(k-1)\alpha_1 u} + e^{-(k+1)\alpha_1 u}]$$

$$- \sum_{k=1}^{N} ka_k^*(\alpha_2)[e^{(k\alpha_2-\alpha_1)u} + e^{-(k\alpha_2+\alpha_1)u}],$$

the derivative of the latter function consequently has at least $2N$ real zeros. When putting

$$dF_1(u)/du = e^{-2\alpha_1 u}F_2(u),$$

we further infer that

$$dF_2(u)/du = \sum_{k=2}^{N} (k^2 - 1^2)\,\alpha_1^2 ka_k^*(\alpha_1)[e^{(k+1)\alpha_1 u} + e^{-(k-1)\alpha_1 u}]$$

$$- \sum_{k=1}^{N} (k^2\alpha_2^2 - \alpha_1^2)\, ka_k^*(\alpha_2)[e^{(k\alpha_2+\alpha_1)u} + e^{-(k\alpha_2-\alpha_1)u}]$$

has at least $2N - 1$ real zeros. In the expression for $dF_2(u)/du$ the coefficient $a_1^*(\alpha_1)$ no more appears. By proceeding this reasoning we can successively eliminate $a_3^*(\alpha_1)$, $a_5^*(\alpha_1)$,..., $a_{N-1}^*(\alpha_1)$, and find a function

$$\frac{dF_N(u)}{du} = \sum_{k\ \text{even}} \left\{ \prod_{m=1}^{N/2} [k^2 - (2m-1)^2]\, \alpha_1^N ka_k^*(\alpha_1)[e^{(k+N-1)\alpha_1 u} + e^{-(k-N+1)\alpha_1 u}] \right\}$$

$$- \sum_{k=1}^{N} \left\{ \prod_{m=1}^{N/2} [k^2\alpha_2^2 - (2m-1)^2\,\alpha_1^2]\, ka_k^*(\alpha_2)[e^{(k\alpha_2+(N-1)\alpha_1)u} + e^{-(k\alpha_2-(N-1)\alpha_1)u}] \right\},$$

which must have at least $N+1$ real zeros. Next we successively eliminate $a_2^*(\alpha_2)$, $a_4^*(\alpha_2)$,..., $a_N^*(\alpha_2)$, starting from

$$dF_N(u)/du = e^{(2\alpha_2+(N-1)\alpha_1)u}F_{N+1}(u).$$

We finally arrive at a function

$$\begin{aligned} dF_{2N}(u)/du = &\sum_{k\text{ even}} \Big\{ \prod_{m=1}^{N/2} [k^2 - (2m-1)^2]\,\alpha_1^N \prod_{m=1}^{N/2} [k^2\alpha_1^2 - (2m)^2\alpha_2^2]\, ka_k^*(\alpha_1) \\ &\times [e^{(k\alpha_1+N\alpha_2)u} + e^{-(k\alpha_1-N\alpha_2)u}]\Big\} \\ &- \sum_{k\text{ odd}} \Big\{ \prod_{m=1}^{N/2} [k^2\alpha_2^2 - (2m-1)^2\,\alpha_1^2]\,\alpha_2^N \\ &\times \prod_{m=1}^{N/2} [k^2 - (2m)^2]\, ka_k^*(\alpha_2)[e^{(k+N)\alpha_2 u} + e^{-(k-N)\alpha_2 u}]\Big\}, \end{aligned} \tag{6.7}$$

which must have at least one real zero.

Consider the first sum in (6.7). The first product (with k even) contains $k/2$ positive and $(N-k)/2$ negative factors; the second product contains, by virtue of condition (6.5), $(k-2)/2$ positive and $(N-k+2)/2$ negative factors. The sign of this first sum consequently is $[(-1)^{N-k-1}\operatorname{sgn} a_k^*]_{k\text{even}} = +1$. Consider next the second sum in (6.7). The first product (with k odd) has, by virtue of condition (6.5), $(k+1)/2$ positive and $(N-k-1)/2$ negative factors; the second product has $(k-1)/2$ positive and $(N-k+1)/2$ negative factors. Hence, the sign of this second sum is $[(-1)^{N-k}\operatorname{sgn} a_k^*]_{k\text{odd}} = -1$. The conclusion is that all terms in (6.7) have the same (positive) sign for all values of u, which contradicts the fact that dF_{2N}/du has at least one real zero. Hence, the lemma is proved for the function $F_e(u)$ and for even N. A completely analogous proof holds for odd N (we then eliminate $a_1^*(\alpha_1)$, $a_3^*(\alpha_1)$,..., $a_N^*(\alpha_1)$, $a_2^*(\alpha_2)$, $a_4^*(\alpha_2)$,..., $a_{N-1}^*(\alpha_2)$).

By considering u as a complex variable and by substituting $e^{\alpha_2 u}$ for z in (6.6.a), we obtain a function of the form $\frac{1}{2}z^{-N}g(z)$, with

$$g(z) = \sum_{k=1}^{N} ka_k^*(\alpha_1)(z^{N+k\alpha_1/\alpha_2} + z^{N-k\alpha_1/\alpha_2}) - \sum_{k=1}^{N} ka_k^*(\alpha_2)(z^{N+k} + z^{N-k}). \tag{6.8}$$

We have proved that the function $g(z)$ has at most $2N$ real zeros. Observe that this is also true for functions of the forms (6.6.a) and (6.8) with arbitrary coefficients $c_{k,1}$, $c_{k,2}$, $k = 1, 2,..., N$, provided the $c_{k,1}$ have alternating signs and $\operatorname{sgn} c_{k,1} = \operatorname{sgn} c_{k,2}$. Since the function (6.6.b) reduces to the form (6.8) (apart from a factor $z^{-N}/2$) after the substitution $e^{i\alpha_2 u} = z$ (with $\alpha_2 \mid u \mid \leqslant \pi$), and since sgn $b_k{}^* =$ sgn $a_k{}^*$, the lemma is also true for $F_t(u)$.

LEMMA C. *If α_1, α_2 satisfy condition* (6.5), *then there exists a constant C, $0 < C < 1$, such that for each constant K, $1 \geqslant K \geqslant C$, the functions*

$$G_e(u) = \sum_{k=1}^{N} a_k(\alpha_1) \sinh \alpha_1 ku - K \sum_{k=1}^{N} a_k(\alpha_2) \sinh \alpha_2 ku + (K-1)u, \quad (6.9.a)$$

$$G_t(u) = \sum_{k=1}^{N} b_k(\alpha_1) \sin \alpha_1 ku - K \sum_{k=1}^{N} b_k(\alpha_2) \sin \alpha_2 ku + (K-1)u, \quad (6.9.b)$$

$$dG(_e u)/du = \sum_{k=1}^{N} ka_k{}^*(\alpha_1) \cosh \alpha_1 ku - 1 - K\left(\sum_{k=1}^{N} ka_k{}^*(\alpha_2) \cosh \alpha_2 ku - 1\right), \quad (6.10.a)$$

$$dG_t(u)/du = \sum_{k=1}^{N} kb_k{}^*(\alpha_1) \cos \alpha_1 ku - 1 - K\left(\sum_{k=1}^{N} kb_k{}^*(\alpha_2) \cos \alpha_2 ku - 1\right), \quad (6.10.b)$$

have at most N positive zeros.

Proof. We choose two values α_1 and α_2 close enough together such that condition (6.5) is satisfied. Then, since $\sum_{k=1}^{N} ka_k{}^* - 1$ is a negative monotonically decreasing function of α (Theorem I), there exists a constant C_1, $0 < C_1 < 1$, such that for each constant K_1, $1 \geqslant K_1 \geqslant C_1$,

$$dG_e(0)/du = \sum_{k=1}^{N} ka_k{}^*(\alpha_1) - 1 - K_1\left(\sum_{k=1}^{N} ka_k{}^*(\alpha_2) - 1\right) > 0. \quad (6.11)$$

By virtue of Theorem I and for α_1 and α_2 close enough together there also exists a constant C_2, $0 < C_2 < 1$, such that for each constant K_2, $1 \geqslant K_2 \geqslant C_2$,

$$\operatorname{sgn}\left[\left(\sum_{k=1}^{N} ka_k{}^*(\alpha_1) \cosh k\alpha_1 - 1\right) - K_2\left(\sum_{k=1}^{N} ka_k{}^*(\alpha_2) \cosh k\alpha_2 - 1\right)\right] = (-1)^N. \quad (6.12)$$

Both inequalities (6.11) and (6.12) are satisfied for constants K such that $1 \geqslant K \geqslant \max(K_1, K_2)$. For such a constant the function (6.10.a) cannot have $N+2$ positive zeros; otherwise its derivative should have $N+2$ nonnegative zeros (one of them being $u = 0$), and its second derivative, which is a function of the type (6.6.a) (the coefficients of which also have alternating signs), $N + 1$ positive zeros, which is impossible by virtue of Lemma b. This shows that the function (6.10.a) has at most N positive zeros. A similar proof holds for the function (6.10.b). Consequently, $G_e(u)$ and $G_l(u)$ have at most $2N + 1$ real zeros, and, hence, at most N positive zeros.

THEOREM J. *d_e and $|d_l|$ are monotonically increasing functions of α.*

Proof. Suppose there exist two values α_1 and α_2, $\alpha_1 \neq \alpha_2$, such that $d_e(\alpha_1) = d_e(\alpha_2)$. Then the function

$$H_e(u) = \sum_{k=1}^{N} a_k(\alpha_1) \sinh \alpha_1 k u - \sum_{k=1}^{N} a_k(\alpha_2) \sinh \alpha_2 k u$$

has at least one zero in each interval $(\bar{e}_i(\alpha_1), \bar{e}_{i+1}(\alpha_1)]$, $i = 1, 2, \ldots, N$, with $\bar{e}_{N+1} = 1$. Furthermore, by virtue of Theorem I, there also lies at least one zero in $(0, \bar{e}_1(\alpha_1)]$. This implies that $H_e(u)$ has at least $N + 1$ positive zeros. By Lemma c this is impossible since $H_e(u) = G_e(u)|_{K=1}$. Hence, d_e is a monotonic function of α. A similar reasoning shows that this is also true for d_l. The fact that $d_e' > 0$ and $|d_l|' > 0$ for small values of α (see Eqs. (4.12) and (4.14)) completes the proof of the theorem.

THEOREM K. *The quantities $\bar{e}_i$, e_i are monotonically increasing functions of α; the quantities $\bar{l}_i$, l_i are monotonically decreasing functions of α.*

Proof. Suppose there exist two values α_1, α_2, $\alpha_1 < \alpha_2$, and an index i such that

$$\bar{e}_i(\alpha_1) < \bar{e}_i(\alpha_2) < \bar{e}_{i+1}(\alpha_2) < \bar{e}_{i+1}(\alpha_1).$$

Consider the function (6.9.a), with $K = d_e(\alpha_1)/d_e(\alpha_2)$. For α_1 chosen close enough to α_2 (which is always possible), K satisfies the condition of Lemma c. Then the function (6.9.a) has at least three zeros in the interval $(\bar{e}_i(\alpha_1), \bar{e}_{i+1}(\alpha_1)]$, while in each other interval $(\bar{e}_j(\alpha_1), \bar{e}_{j+1}(\alpha_1)]$ there is at least one zero. Therefore, this function has at least $N + 2$ positive zeros, in contradiction with Lemma c. In an analogous way we can show that e_i, $\bar{l}_i$, and l_i are monotone in α. Since $\lim_{\alpha\to\infty} \bar{e}_i = \lim_{\alpha\to\infty} e_i = 1$ (Theorem F), the $\bar{e}_i$ and e_i are increasing functions of α. This implies that (Theorem D) $\bar{C}_{e,i,2} > 0$, $C_{e,i,2} > 0$, and that $\bar{C}_{l,i,2} < 0$, $C_{l,i,2} < 0$, which reveals that $\bar{l}_i$ and l_i are decreasing functions of α.

It may be remarked here that Knight and Newbery [10] conjecture analog monotonicity properties for integration nodes appearing in quadrature rules based on exponential and trigonometric interpolation.

THEOREM L. *$|\epsilon_e|$ is a monotonically decreasing function of α; $|\epsilon_l|$ is a monotonically increasing function of α.*

Proof. Suppose there are two values α_1 and α_2, $\alpha_1 < \alpha_2$, such that $\epsilon_e(\alpha_1) = \epsilon_e(\alpha_2)$. Then $\alpha_2^{2N} d_e(\alpha_1) = \alpha_1^{2N} d_e(\alpha_2)$. Consider again the function (6.9.a), with $K = (\alpha_1/\alpha_2)^{2N}$; K satisfies the condition of Lemma c if α_1 is chosen close enough to α_2. This function has at least one zero in each interval $(\bar{e}_i(\alpha_1), \bar{e}_{i+1}(\alpha_1)]$, $i = 1, 2, \ldots, N-1$, and at least two in $(\bar{e}_N(\alpha_1), 1]$. This leads to the contradiction that $G_e(u)$ should have at least $N+1$ positive zeros. Hence, ϵ_e is monotonic. Since $q_{2N+2} < 0$ (Theorem E), $|\epsilon_e|$ is decreasing for small values of α. Consequently this is true for all α.

In a similar way we can show that ϵ_t is monotone. Here, however, there should be at least one zero of the function (6.7.b) in each interval $(\bar{t}_i(\alpha_1), \bar{t}_{i+1}(\alpha_1)]$, $i = 0, 1, \ldots, N$, with $\bar{t}_0 = 0$ and $\bar{t}_{N+1} = 1$. Since $\operatorname{sgn} r_{2N+2} = \operatorname{sgn} r_{2N}$ (Theorem E), $|\epsilon_t|$ increases for small values of α. Consequently, this is true for all α.

7. Concluding Remarks

The results of the last three sections admit the following conclusions:

(i) Ignoring the unestimable factor $L_{2N+1}[f(\xi)]$ in the remainders of the interpolation formulas under consideration, we may say that the exponential interpolation type is always "better" than the polynomial type, while the trigonometric interpolation type is always "worse" (Theorems C and L).

(ii) The inequalities $t_i < p_i < e_i$, $i = 1, 2, \ldots, N$, hold for all $\alpha > 0$ (Theorems B and K); this means that the optimal points for the exponential interpolation type have a tendency of being located near the end points of the interpolation interval, while the optimal points for the trigonometric interpolation type have a tendency of lying near the midpoint of that interval.

In view of these conclusions it might sometimes be possible to decide what interpolation type is most appropriate to a given table of data $\{x_i, f(x_i)\}$, $i = 0, 1, \ldots, n$, to be interpolated.

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